Some properties on four dimensional Walker-S-manifolds

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Ecole mathématique Africaine du 15 au 27 juillet 2025 Brazzaville, Congo
Arithmétique, Géométrie, Calcul Formel et Applications
AGCFA

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Introduction

The study of sectional curvature has long been one of the most crucial subjects since it provides information about certain characteristics of manifolds. In the framework of Riemannian geometry, almost S-manifolds (of dimension 2n+s) represent a natural generalization of contact and Sasaki manifolds , Such manifolds have been extensively studied by several authors and from different perceptives.

Objective of the talk

In this presentation we study Walker manifold with neutral signature in dimension four endowed with a locally conformal change s-structure.

S-manifold and locally conformal s-manifolds

S-manifolds

A $(2n+s)$-dimensional Riemannian manifolds (M, g) endowed with an ϕ-structure where ϕ is a (1,1)-tensor field of rank $2n$ satisfying $\phi^3 + \phi = 0$ is a metric ϕ-manifolds if there exists s global vector fields $\xi_1, \cdots, \xi_s$ on TM called structure vector fields such that, their respective dual 1-forms $\eta_1, \cdots, \eta_s$ verifies

$$\phi\xi_i = 0,\ \eta^i \circ \phi = 0,\ \phi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i;$$

$$g(\phi X, \phi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y), \tag{1}$$

for any $X, Y \in \mathfrak{X}(M)$ and $i = 1, \cdots, s$.

Let Φ be the 2-form on M defined by $\Phi(X, Y) = g(X, \phi Y)$ for any $X, Y \in \mathfrak{X}(M)$, then it is clear that $\eta_1 \wedge \cdots \wedge \eta_s \wedge \Phi^n$ is a volume form therefore M is orientable. The Nijenhuis tensor N_ϕ is then given by

$$N_\phi(\cdot, \cdot) = [\phi, \phi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i.$$

Hence, M is normal if N_ϕ vanishes
A metric ϕ-manifold is said to be a K-manifold if it is normal and $d\Phi = 0$. A K-manifold is called an S-manifold if $\Phi = d\eta_i$, for any $i \in I \subset \mathbb{N}$. It A necessary and sufficient condition for a K-manifold M to be an S-manifold is

$$(\nabla_X \phi)Y = \sum_{i=1}^{s} \{g(\phi X, \phi Y)\xi_i + \eta_i(Y)\phi^2 X\}, \tag{2}$$

where ∇ is the Levi-Civita connection of g. By a direct calculation we have

$$\nabla_X \xi_i = -\phi X,\ X \in \mathfrak{X}(M) \tag{3}$$

Properties

for an S-manifold. As immediate properties we have:

i) The orthogonal splitting of TM as $TM = Im(\phi) \oplus \ker(\phi)$.

ii) $\nabla_{\xi_\alpha}\phi = 0$, $\alpha \in \{1, ..., s\}$

iii) $\nabla_{\xi_\alpha}\xi_\beta = 0$, $\alpha, \beta \in \{1, ..., s\}$

We shall denote $D = Im(\phi)$ and $D^\perp = \ker(\phi) = \{\xi_1, ..., \xi_s\}$.

Locally conformal S-manifolds

Let M be an indefinite globally framed f-manifold and (ϕ, ξ_i, η^i, g) its globally framed f-structure. The manifold $(M, \phi, \xi_i, \eta^i, g)$, $i \in \{1, ..., s\}$ is said to be locally conformal S-manifold if M has an open covering $\{U_t\}_{t\in I}$ endowed with smooth functions
$\sigma_t : U_t \longrightarrow \mathbb{R}$ such that over each U_t the globally framed f-structure $(\phi_t, \xi_i^t, \eta_t^i, g_t)$ defined by

$$\phi_t = \phi, \;\; \eta_t^i = \exp(-\sigma_t)\eta^i, \;\; \xi_i^t = \exp(\sigma_t)\xi_i, \;\; g_t = \exp(-2\sigma_t)g \tag{4}$$

is S-manifold. That is M is called locally conformal S-Manifold if for each t, $(U_t, \phi_t, \xi_i^t, \eta_t^i, g_t)$ is a S-Manifold.

It is well known that for any conformal transformation σ_t on open covering $\{\sigma_t\}$ the equality

$$\nabla^t_X Y = \nabla_X Y - \omega(X)Y - \omega(Y)X + g(X, Y)B, \tag{5}$$

holds. Where $\omega = d\sigma$ called Lee 1-form and B its g-dual vector field, that is $\omega(X) = g(B, X)$.

Theorem

In a locally conformal $s-$manifold one has

$$\begin{aligned} (\nabla_X \phi)Y &= \exp(-\sigma)[g(\phi X, \phi Y)\bar{\xi} + \bar{\eta}(Y)\phi^2 X] + \omega(\phi Y)X - \omega(Y)\phi X \\ &- g(X, \phi Y)B + g(X, Y)\phi B, \end{aligned} \tag{6}$$

where $\bar{\eta} = \sum_{i=1}^{s} \eta^i$ and $\bar{\xi} = \sum_{i=1}^{s} \xi_i$, which implies that $\nabla^t_X \xi^t_i = -\phi X$ and that $ker(\phi)$ is an integrable flat distribution. We remark that an indefinite $S-$manifold is never flat since $K^t(X, \xi^t_i) = \varepsilon_i$ for any $X \in D_p$.

Considering the tensor fields h on M by $2h_i X = (\nabla_{\xi_i}\phi)X - \eta^i(X)\phi B$ Clearly, we can see that in each open covering U_t, $hX = -e^{-\sigma_t}\phi X$, hence it is easy to check that $h_i\xi_i = 0$, $h\phi = -\phi h$ and $trace(h_i) = 0$. Since $h_1\xi_2 = h_2\xi_1 = 0$ then $B = \nabla_{\xi_1}\overline{\xi} + \omega(\overline{\xi})\xi_1 = \nabla_{\xi_2}\overline{\xi} + \omega(\overline{\xi})\xi_2$.

Corollary

Let (M, ξ_i, η^i, g) be a locally conformally s-manifold. The Lee vector field belongs to the distribution $D^\perp$.

Theorem

Let $(M, \phi, \xi_i, \eta_i, g)$ an indefinite globally framed f-manifold. The following statements are equivalentes

(i) The manifold $(M, \phi, \xi_i, \eta_i, g)$ is locally conformally s-manifold

(ii) There are functions f^α, $\alpha \in \{1, ..., s\}$ such that the Lee form satisfies $\omega = f_\alpha \eta^\alpha$

(iii) $(\nabla_X\phi)Y =$
$-e^{-\sigma}(g(\phi^2 X, Y)\overline{\xi} - \overline{\eta}(Y)\phi^2 X) + f^\alpha(g(\phi X, Y)\xi_\alpha - \eta_\alpha(Y)\phi X)$

Four dimensional Walker manifolds

Let (M, g) be an n-dimensional semi-Riemannian manifold, and ∇ the Levi-Civita connection on M. If D is a distribution on M, naturally $\mathcal{D}^{\perp}$, $\mathcal{D} + \mathcal{D}^{\perp}$ and $RadTM = \mathcal{D} \cap \mathcal{D}^{\perp}$ are also distributions on M and $\mathcal{D} + \mathcal{D}^{\perp}$ need not be equal to TM and $RadTM$ need not be trivial since this depends upon the degree of nullity of $\mathcal{D}$.

Definition

The distribution $\mathcal{D}$ is parallel with respect to the Levi-Civita connection on (M, g) if for any vector field X of $\mathcal{D}$, ∇X takes values in $\mathcal{D}$. Moreover, if $\mathcal{D}$ is totally null the manifold M is called Walker manifold.

Theorem

[1] A canonical form for a $2n$ dimensional pseudo-Riemannian manifold M admitting a parallel field of null n planes D is given by the metric tensor:

$$(g_{ij}) = \begin{bmatrix} 0 & Id_n \\ Id_n & B \end{bmatrix}$$

where Id_n is the $n \times n$ identity matrix and B is a symmetric $n \times n$ matrix whose entries are functions of the coordinates $(x_1, \cdots, x_{2n})$.

According to the above theorem, the metric of four-dimensional Walker manifold is expressed in local coordinate system (x, y, z_1, z_2) by

$$(g_{abc}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{bmatrix} \tag{7}$$

where a, b, c are functions depending on x, y, z_1 and z_2.

Proposition

Let (M, g) be a Walker manifold. The conformal transformation of g given by the equality (4) is not a Walker metric. There is a globally linear connexion $\widetilde{\nabla}$ of torsion free obtained by gluying up the connexion ∇^t of (5) and satisfies the relation

$$\widetilde{\nabla} g = 2\omega \otimes g. \tag{8}$$

Let (M, g) be a four-dimensional Walker manifold where the metric g is given by (7) and $\omega = d\sigma$ be the Lee 1-form which satisfies (8).
Using the christofeil formulas, the non vanishing components of the connexion $\widetilde{\nabla}$ are expressed by

$$
\begin{aligned}
\widetilde{\nabla}_{\partial x}\partial x &= -2\sigma_1\partial x,\\
\widetilde{\nabla}_{\partial x}\partial y &= -\sigma_2\partial x - \sigma_1\partial y,\\
\widetilde{\nabla}_{\partial x}\partial z_1 &= (\sigma_4 - c\sigma_1 - b\sigma_2 + 1/2c_1)\partial y - \sigma_2\partial z_2,\\
\widetilde{\nabla}_{\partial x}\partial z_2 &= (-\sigma_4 + 1/2c_1)\partial x - \sigma_1\partial z_2,\\
\widetilde{\nabla}_{\partial y}\partial y &= -2\sigma_2\partial y,\ \widetilde{\nabla}_{\partial y}\partial z_1 = 1/2a_2\partial x + (\sigma_3 + 1/2c_2)\partial y - \sigma_2\partial z_1,\\
\widetilde{\nabla}_{\partial y}\partial z_2 &= (\sigma_3 - a\sigma_1 - c\sigma_2)\partial x + (1/2b_2 - c_1\sigma_1 - b\sigma_2)\partial y + \sigma_1\partial z_1,\\
\widetilde{\nabla}_{\partial z_1}\partial z_1 &= ((\sigma_3 + a\sigma_1 + c\sigma_2)a + 1/2a_3)\partial x + (\sigma_4 - c\sigma_1 - b\sigma_2)\partial y\\
&\quad + (\sigma_1 - 2\sigma_3 - 1/2a_1)\partial z_1 + (\sigma_2 a - 1/2a_2)\partial z_2,\\
\widetilde{\nabla}_{\partial z_1}\partial z_2 &= \left[(\sigma_3 - a\sigma_1 - c\sigma_2)c + 1/2(ac_1 + cc_2 - c_3)\right]\partial x\\
&\quad + \left[(\sigma_4 - c\sigma_1 + b\sigma_2)c + 1/2(cc_1 - bc_2 + c_3)\right]\partial y\\
&\quad + (c\sigma_1 - \sigma_4 + 1/2c_1)\partial z_1 + (\sigma_3 c - \sigma_3 - 1/2c_2)\partial z_2,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\nabla}_{\partial z_2}\partial z_2 = {} & [(\sigma_3 - a\sigma_1 - c\sigma_2)b + 1/2(ab_1 + cb_2 + 2c_4 - b_4)]\,\partial x \\
& + [(\sigma_3 - a\sigma_1 - c\sigma_2)c + 1/2(cb_1 + bb_2 + b_4)]\,\partial y \\
& + (\sigma_1 b - 1/2b_1)\partial z_1 - (\sigma_2 - 2\sigma_4 - 1/2b_2)\partial z_2 \qquad (9)
\end{aligned}
$$

The components of the connection ∇ are easily obtained by vanishing of the function σ.
The Weyl connection $\widetilde{\nabla}$ is not symmetrical over all of its arguments. Moreover it is not a Levi-Civita one but it coincides with the Levi-Civita connection of $\tilde{g}$ on $\ker(\omega)$. Thus the leaves of the distribution $\ker(\omega)$ are integral submanifolds of M.

The following tensors are well known

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and

$$R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

.

In the next section we give an example of four-dimensional locally conformal walker S-manifold. We recall that the structure (g,ϕ,ξ^1,ξ^2) is locally conformal S-manifold on M if $d\widetilde{\eta}^1 = d\widetilde{\eta}^2 = \Phi$ and $\widetilde{N}_\phi = 0$ that is ϕ is normal. To this end, we make use of a orthonormal basis given in [1]

$$\begin{aligned} E_1 &= c\partial x + \frac{1}{2}(1-b)\partial y + \partial z_2, \; E_3 = \frac{1}{2}(1-a)\partial x + \partial z_1 \\ E_2 &= -c\partial x - \frac{1}{2}(1+b)\partial y + \partial z_2, \; E_4 = -\frac{1}{2}(1+a)\partial x + \partial z_1 \end{aligned} \tag{10}$$

Example

Let (M^4, g) be a Walker manifold and $\exp(\sigma)\{E_1, E_2, E_3, E_4\}$ be the $\widetilde{g}$-orthonormal basis (10) with $\widetilde{g} = \exp(-2\sigma)g$, setting $\xi_1 = e^\sigma E_3$ and $\xi_2 = e^\sigma E_4$, the tensor field ϕ given by

$$\phi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

A straightforward calculation gives $\phi^2 E_1 = -E_1 + \eta^1 \otimes \xi^1 + \eta^2 \otimes \xi^2$ and $\phi^2 E_2 = -E_1 + \eta^1 \otimes \xi_1 + \eta^2 \otimes \xi_2$. Therefore, in terms of canonical basis $\{\partial x, \partial y, \partial z_1, \partial z_2\}$ one gets

$$\phi = \begin{bmatrix} 0 & 0 & 0 & -c \\ 0 & b & 0 & \frac{b^2+1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -b \end{bmatrix} \tag{12}$$

Example

Then the second form fundamental is locally given by

$$\widetilde{\Phi} = be^{-2\sigma} dz_2 \wedge dy. \tag{13}$$

The condition $d\widetilde{\eta}^i = \widetilde{\Phi}$ for any $i = 1, 2$ holds if we have the following partial derivative equations

$$\sigma_1 = \sigma_2 = \sigma_4 = a_2 = 0 \text{ and } b = -e^{\sigma} c_2.$$

Where $\eta^1 = dx + \frac{1}{2}(1+a)dz_1 + cdz_2$ and $\eta^1 = dx - \frac{1}{2}(1-a)dz_1 + cdz_2$. Clearly the conformal function σ depends only z_1. Thus the Lee one-forme also its Lee vector field are obtained by

$$\omega = \sigma_3 dz_1 \text{ and } B = (\sigma_3 - ka)\partial x - kc\partial y + k\partial z_1,\ k \in \mathbb{R}. \tag{14}$$

We easily verify that $h_i\partial x = 0$,
$h_i\partial y = \{e^{-\sigma}(\overline{\eta}(\xi_i) - 1) + 2bf^1\}\partial y - 4f^1\partial z_2$, $h_i\partial z_1 = 0$, for $i = 1$, 2 and $h_1\partial z_2 = (a+1)e^{-\sigma}\partial x - 2ce^{-\sigma}\partial z_1$,

Example

$h_2\partial z_2 = (1-a)ce^{-\sigma}\partial x + 2ce^{-\sigma}\partial z_1 - 2e^{-\sigma}\partial z_2.$
Putting $\sigma = \ln z_1$, defined on the open neighbour
$U = \{(x, y, z_1, z_2), |z_1 > 0 \in \mathbb{R}\}$. The Lee form is $\omega = \frac{1}{z_1}dz_1$ and B its g-dual vector field obtained by $B = (\frac{1}{z_1} - a)\partial x - c\partial y + \partial z_1$. From equality (9), one gets the non-vanishing components of the connection, as follows $\widetilde{\nabla}$,
$\widetilde{\nabla}_{\partial x}\partial x = -2\frac{1}{x}\partial x, \widetilde{\nabla}_{\partial x}\partial y = -\frac{1}{x}\partial y, \widetilde{\nabla}_{\partial x}\partial z_1 = (-c\frac{1}{x} + 1/2c_1)\partial y, \widetilde{\nabla}_{\partial x}\partial z_2 = (1/2c_1)\partial x - \frac{1}{x}\partial z_2, \widetilde{\nabla}_{\partial y}\partial y = 0,\ \widetilde{\nabla}_{\partial y}\partial z_1 = (1/2c_2)\partial y, \widetilde{\nabla}_{\partial y}\partial z_2 = (-a\frac{1}{x})\partial x + (1/2b_2 - c_1\frac{1}{x})\partial y + \frac{1}{x}\partial z_1, \widetilde{\nabla}_{\partial z_1}\partial z_1 = ((\frac{1}{x})a^2)\partial x + (-c\frac{1}{x})\partial y + (\frac{1}{x})\partial z_1, \widetilde{\nabla}_{\partial z_1}\partial z_2 = [(-a\frac{1}{x})c + 1/2(ac_1 + cc_2)]\partial x + [(-c\frac{1}{x})c + 1/2(cc_1 - bc_2)]\partial y + (c\frac{1}{x} + 1/2c_1)\partial z_1 + (-1/2c_2)\partial z_2, \widetilde{\nabla}_{\partial z_2}\partial z_2 = [(-a\frac{1}{x})b + 1/2(ab_1 + cb_2 + 2c_4 - b_4)]\partial x + [(-a\frac{1}{x})c + 1/2(cb_1 + bb_2 + b_4)]\partial y + (\frac{1}{x}b - 1/2b_1)\partial z_1 + 1/2b_2)\partial z_2.$

Some properties of ϕ-sectional curvature

Let's define the symmetric tensor field P by

$$P(X,Y) = (\nabla_X \omega)Y + \omega(X)\omega(Y) - \frac{1}{2}|B|^2 g(X,Y), \tag{15}$$

then

$$P(\bar{\xi},\bar{\xi}) = \bar{\xi}\omega(\bar{\xi}) + \frac{1}{2}|B|^2, \ \ \mathrm{tr}P = \mathrm{div}B - \frac{1}{2}(2n-1)|B|^2 \tag{16}$$

Hence, the Riemannian curvatures are related by the means of (**??**) as follows

$$\begin{aligned} e^{2\sigma}\widetilde{R}(X,Y,Z,W) = {} & R(X,Y,Z,W) + g(X,W)P(Y,Z) - g(X,Z)P(Y,W) \\ & + g(Y,Z)P(X,W) - g(Y,W)P(X,Z) \end{aligned} \tag{17}$$

The Ricci and the scalar curvature are known respectively by

$$S(X,Y) = \sum_{i=1}^{2n+s} R(E_i,X,Y,E_i) \text{ and } \tau = \sum_{i=1}^{2n+s} S(E_i,E_i) \tag{18}$$

(see [10])

Recall that $K_p(\pi) = \dfrac{R_p(X, Y, X, Y)}{\Delta_p(\pi)}$ where
$\Delta_p(\pi) = g_p(X, X)g_p(Y, Y) - g_p(X, Y)^2 \neq 0.$

Let M be a $2n + s$-dimensional *gff*-manifolds. A 2-plane π of T_pM is said to be $\phi-$holomorphic plane if π is orthogonal to $D^{\perp} = \mathrm{span}\{\xi_1, \cdots, \xi_s\}$ and $\phi(\pi) = \pi$. It is well known that a pointwise constant ϕ-holomorphic sectional curvature $K(\pi)$ does not dependent on the choice of the ϕ-holomorphic plane π of T_pM and the function H defined by $H(p) = K(\pi)$ where $p \in M$ is called ϕ-holomorphic sectional curvature of M. Thus M is of constant ϕ-holomorphic sectional curvature c if the function H is contant and identically equal to c

Pointwise constante ϕ-sectional curvature

The (0,4)-type Riemannian curvature for a pointwise constant ϕ-sectional curvature c as follows

$$\begin{aligned} R(X,Y,Z,W) = &-\frac{H+3\varepsilon}{4}\{g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)\} \\ &- \frac{H-\varepsilon}{4}\{\Phi(W,X)\Phi(Z,Y) - \Phi(Z,X)\Phi(W,Y) \\ &+ 2\Phi(X,Y)\Phi(W,Z)\} - \{\overline{\eta}(W)\overline{\eta}(X)g(\phi Z,\phi Y) \\ &- \overline{\eta}(W)\overline{\eta}(Y)g(\phi Z,\phi X) + \overline{\eta}(Y)\overline{\eta}(Z)g(\phi W,\phi X) \\ &- \overline{\eta}(Z)\overline{\eta}(X)g(\phi W,\phi Y)\} \end{aligned} \tag{19}$$

where $\varepsilon = \sum_i^s \varepsilon_i$ (see [7]).

lemma

Let $(M, \xi_i, \eta^i, \phi, g)$ be an indefinite locally conformal $S-$manifold of pointwise $\phi-$sectional curvature $K_p(\pi)$. Then the sectional curvature $K_p^t(\pi)$ on U_t satisfies

$$K_p^t(\pi) = K_p(\pi) + 3\sum_{i=1}^{s} \varepsilon_i (f^i)^2 \tag{20}$$

where π is any $2-$plane $\{X, \phi X\}$ orthogonal to $D^{\perp} = ker\phi$.

Consequently in each U_t, from the equation (20) the ϕ-sectional curvature is also constant. Since ω is closed then after some computations one gets

$$\begin{aligned} P(X,Y) = &\sum_{i=1}^{s}((f^i)^2 + (f^i)')\eta^i(X)\eta^i(Y) + \sum_{i=1}^{s} f^i(g(h_i X, \phi Y) \\ &+ e^{-\sigma}\Phi(X,Y)) + \sum_{i=1}^{s}(f^i)^2(\Phi(\phi X, Y) - \frac{1}{2}g(X,Y)) \end{aligned} \tag{21}$$

Then the following Theorem holds

Theorem

Let $(M, \phi, \xi_i, \eta^i, g)$ be an indefinite locally conformal $S-$manifold. Then the $\phi-$sectional curvature c is pointwise constant, $c \in C^\infty(M)$, if and only if the Riemannian $(0,4)-$type curvature tensor field R is given by

$$
\begin{aligned}
R(X,Y,Z,W) = {} & -e^{-2\sigma}[\frac{H + 3\sum_{i=1}^{s}(\varepsilon + \varepsilon_i (f^i)^2)}{4}\{g(\phi Y, \phi Z)g(\phi X, \phi W) \\
& - g(\phi X, \phi Z)g(\phi Y, \phi W)\} \\
& - \frac{H + 3\sum_{i=1}^{s}\varepsilon_i (f^i)^2 - \varepsilon}{4}\{\Phi(W,X)\Phi(Z,Y) \\
& - \Phi(Z,X)\Phi(W,Y) + 2\Phi(X,Y)\Phi(W,Z)\} \\
& - \{\overline{\eta}(W)\overline{\eta}(X)g(\phi Z, \phi Y) \\
& - \overline{\eta}(W)\overline{\eta}(Y)g(\phi Z, \phi X) + \overline{\eta}(Y)\overline{\eta}(Z)g(\phi W, \phi X) \\
& - \overline{\eta}(Z)\overline{\eta}(X)g(\phi W, \phi Y)\}] \\
& - g(X,W)[\sum_{i=1}^{s}((f^i)^2 + (f^i)')\eta^i(Y)\eta^i(Z)
\end{aligned}
$$

suite theorem

$$+\sum_{i=1}^{s} f^i(g(h_i Y, \phi Z) + e^{-\sigma}\Phi(Y,Z))$$
$$+\sum_{i=1}^{s}(f^i)^2(\Phi(\phi Y, Z) - \frac{1}{2}g(Y,Z))]$$
$$+ g(X,Z)[\sum_{i=1}^{s}((f^i)^2 + (f^i)')\eta^i(Y)\eta^i(W)$$
$$+\sum_{i=1}^{s} f^i(g(h_i Y, \phi W) + e^{-\sigma}\Phi(Y,W))$$
$$+\sum_{i=1}^{s}(f^i)^2(\Phi(\phi Y, W) - \frac{1}{2}g(Y,W))]$$

suite theorem

$$
\begin{aligned}
&- g(Y,Z)[\sum_{i=1}^{s}((f^i)^2 + (f^i)')\eta^i(X)\eta^i(W) \\
&+ \sum_{i=1}^{s} f^i(g(h_i X, \phi W) + e^{-\sigma}\Phi(X,W)) \\
&+ \sum_{i=1}^{s}(f^i)^2(\Phi(\phi X, W) - \frac{1}{2}g(X,W))] \\
&+ g(Y,W)[\sum_{i=1}^{s}((f^i)^2 + (f^i)')\eta^i(X)\eta^i(Z) \\
&+ \sum_{i=1}^{s} f^i(g(h_i X, \phi Z) + e^{-\sigma}\Phi(X,Z)) \\
&+ \sum_{i=1}^{s}(f^i)^2(\Phi(\phi X, Z) - \frac{1}{2}g(X,Z))] \qquad (22)
\end{aligned}
$$

Now, we have the following results

Theorem

Let $(M, \phi, \xi_i, \eta^i, g)$ be an Walker locally conformal $S-$manifold, of pointwise constant $\phi-$sectional curvature H. The diagonal enties of the Ricci-curvature matrix and the scalar curvature are given by

1. $$S(\xi_1, \xi_1) = (f^1)^2 + 2(f^1)' - 2(f^2)^2 - (f^2)' \tag{23}$$

2. $$S(\xi_2, \xi_2) = (f^1)' - \frac{3}{2}(f^2)^2 - (f^2)' \tag{24}$$

3. $$S(E_1, E_1) = e^{-2\sigma}\left(\frac{H + 3\sum_{i=1}^{s}(\varepsilon + \varepsilon_i(f^i)^2)}{4} + 3\frac{H + 3\sum_{i=1}^{s}\varepsilon_i(f^i)^2 - \varepsilon}{4}\right) + (f^1)^2 + (f^1)' + (f^2)^2 + (f^2)' \tag{25}$$

Suite Theorem

4.

$$S(E_2,E_2) = e^{-2\sigma}\Big(\frac{H + 3\sum_{i=1}^{s}(\varepsilon + \varepsilon_i (f^i)^2)}{4} - \frac{H + 3\sum_{i=1}^{s}\varepsilon_i (f^i)^2 - \varepsilon}{4}\Big) - 2(f^1)^2 + (f^1)' - 2(f^2)^2 + (f^2)' \qquad (26)$$

We then deduce the scalar curvature tensor with respect to the basis $\{E_1, E_2, \xi_1, \xi_2\}$.

$$\tau = 2e^{-2\sigma}\Big(\frac{H + 3\sum_{i=1}^{s}(\varepsilon + \varepsilon_i (f^i)^2)}{4} + \frac{H + 3\sum_{i=1}^{s}\varepsilon_i (f^i)^2 - \varepsilon}{4}\Big) + 5(f^1)' - \frac{9}{2}(f^2)^2 \qquad (27)$$

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Acknowledgement

Thanks a lot