Renormalization of circle maps with a flat piece

TANGUE NDAWA Bertuel
University of Ngaoundere

École mathématique africaine (EMA) - Brazzaville 2025
"Arithmétique, Géométrie, Calcul Formel et Applications (AGCFA)"

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Stability of the Solar System (SSS) - Dynamic System (DS).

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➦ 1880, **Fuch** $DS_{\text{Fuch}}(DE)$:=$DS_{\text{SR}}(DE)$;

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Some Differential Equations haven't an analytical solution.

Differential Equation Qualitative study (first return map, Rotation Number):=Dynamic System.

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Dynamic system: Definition and examples

Dynamic System:= (X, G, F)
A important class: (X, G, f); where, $G = \mathbb{N}$ and

$$F : (x, n) \in X \times G \longmapsto f^n(x); \quad (1.1)$$

$(\mathcal{S}^1, \mathbb{N}, f)$ illustrates a little more the problem of stability of the solar system and many others.

$$K_f := \mathcal{S}^1 \setminus \cup_{i \geq 0} f^{-i}(U_f) \quad (1.2)$$

1. What is the qualitative description of K_f (“Geometry”)?
2. What is the size occupied by K_f (Hausdorff Dimension)?
3. What is the good tools to study it (Renormalization)?
4. When K_f and K_g have the same size; description (Rigidity)?

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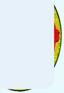

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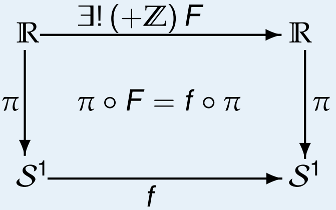
Rotation number of continuous circle map of degree one



$$deg(f) := F(x+1) - F(x) \tag{1.3}$$

if $deg(f) = 1$, then

$$\rho(f) := \lim_{n\to\infty} \frac{F^n(x) - x}{n} (mod\, 1). \tag{1.4}$$

does not depend on x and F.

Assumption

$$\rho(f) = [a_0 a_1 \cdots] \notin \mathbb{Q}. \tag{1.5}$$

Denominators of the convergents of $\rho(f)$

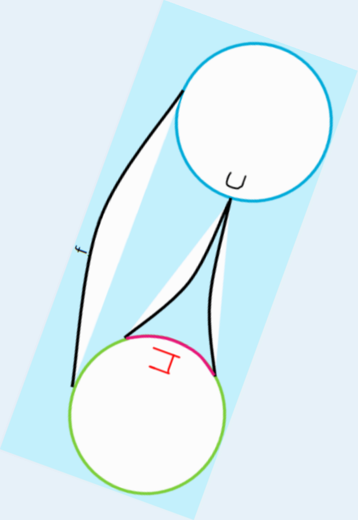
$$q_1 = 1,\ q_2 = a_1 \ \text{ and } q_{n+1} = a_n q_n + q_{n-1},\ n \geq 3. \tag{1.6}$$

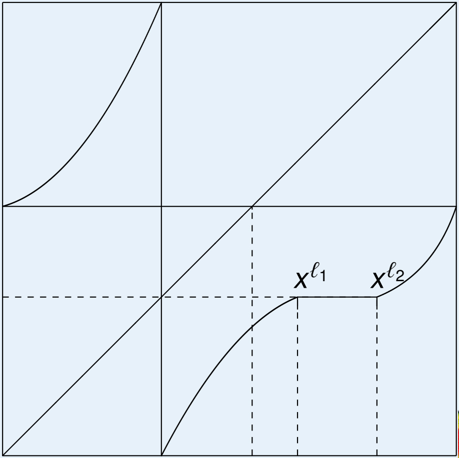
The class of functions $\mathscr{L}$: Description

$\ell_1, \ell_2 \geq 1$. $f \in \mathscr{L}$ if there exist $U := (a, b)$, $c \in \mathcal{S}^1$ and $0 < \epsilon <<$, such that

1. $f(U) = c$.
2. $f|_{\mathcal{S}^1 \setminus \overline{U}}$ is a C^3-diffeomorphism.
3. $$f|_{[b, b+\epsilon]} \approx h_r((x-b)^{\ell_2}); \tag{1.7}$$
   and
   $$f|_{[a-\epsilon, a]} \approx h_l((x-a)^{\ell_1}); \tag{1.8}$$
   $$h_l(x) = h_r(x) = x. \tag{1.9}$$

The class of functions $\mathscr{L}$: Representation





Presentation plan

I Geometry

II Renormalization

Geometry

“Geometry”: qualitative description

$$K_f := \mathcal{S}^1 \setminus \cup_{i\geq 0} f^{-i}(U_f) \tag{2.1}$$

$$\underline{i} := f^{-i}(U_f). \tag{2.2}$$

$$\alpha_n := \frac{|(\underline{-q_n}, \underline{0})|}{|[\underline{-q_n}, \underline{0})|}. \tag{2.3}$$

$$\begin{cases} \text{Degenerate “Geometry” (DG): } \alpha_n \to 0 \\ \text{Bounded “Geometry” (BG): } \alpha_n > K > 0 \end{cases} \tag{Def}$$

$$Sf(x) := \frac{D^3 f(x)}{Df(x)} - \frac{3}{2}\left(\frac{D^2 f(x)}{Df(x)}\right)^2 < 0; \quad \forall\, x,\; Df(x) \neq 0. \tag{A-DG}$$

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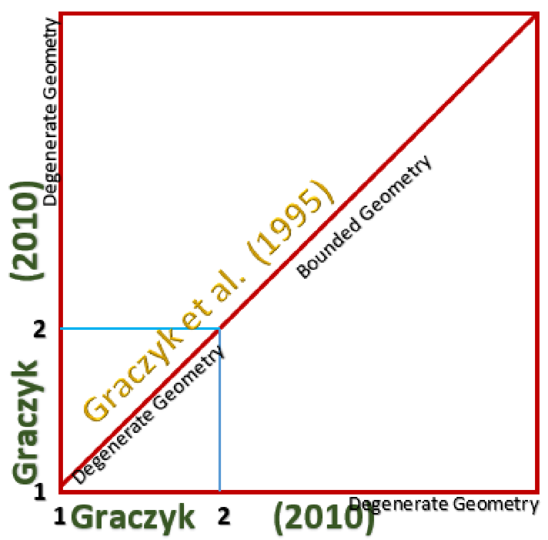
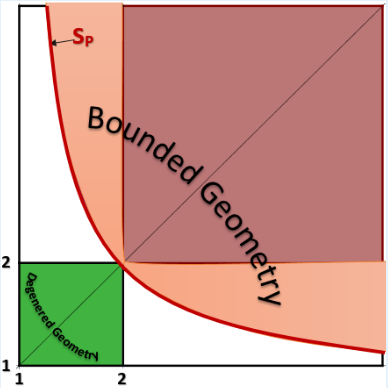
Before our Results



Our Contributions



Theorem 1 (B. TANGUE NDAWA (2020), L. PALMISANO & B. TANGUE NDAWA (2021))

1. *DG:* $(\ell_1, \ell_2) \in [1,2]^2$ *with* ***(A-DG)****;*
2. *BG:* $(\ell_1, \ell_2) \in [2,\infty)^2 \setminus \{(2,2)\}$ *with* ***(A-BG)*** *;*
3. *BG:* $\rho = [abab\cdots]$ $a, b \in \mathbb{N}^*$.

$$t_i(x) = (1 - \ell_i^{-x})(\ell_i - 1)^{-1}. \qquad (2.4)$$

$$\sqrt{(\ell_1^{-b} - \ell_2^{-a})^2 + (t_1(b)t_2(a) + 2(\ell_1^{-b} + \ell_2^{-a}))t_1(b)t_2(a) +} \\ + t_1(b)t_2(a) + \ell_1^{-b} + \ell_2^{-a} - 2 < 0 \qquad I_{a,b}(\ell_1, \ell_2)$$

Renormalization

Renormalization: Story

- Quantum Field Theory (QFT): Dirac (1927); Before, Born, Jordan and Heisenberg
- **inconsistencies (incomprehension) mathematics**: $\infty <<$
- Renormalization (introduction): resizing the variables (masses) and constants (1945-1955): Behe, Feynman, Schwingert, Dyson, Nicolaï Shirkov
- Renormalization Kenneth Wilson (1982) Shirkov (1982-1984)

The renormalization can be defined as the microscopic study of the "Geometry".

- Unimodal Maps: de Melo and Van Strien (1993), etc.

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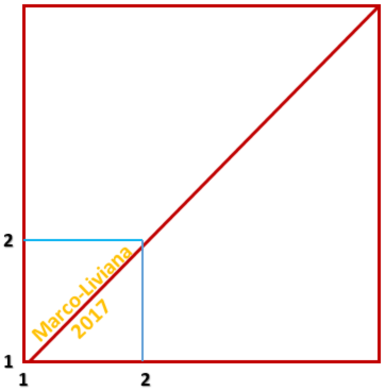
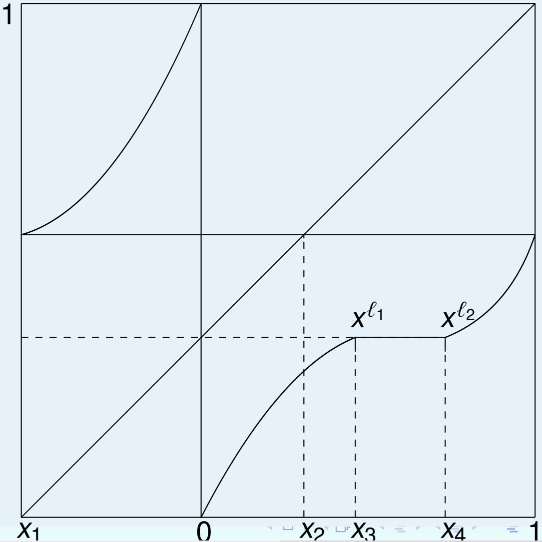
Result of Marco and Liviana



Class of renormalization map

$$\rho(f) = \frac{1+\sqrt{5}}{2} = [111\cdots] := 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}} = \text{ Nombre d'or.}$$



The class of functions

$$\Sigma^X = \{(x_1, x_2, x_3, x_4, s) \in \mathbb{R}^5 | x_1 < 0,\ x_3 < x_4 < 1,\ 0 < x_2,\ s < 1\}, \quad (3.1)$$

$$f := (x_1, x_2, x_3, x_4, s, \varphi, \varphi^l, \varphi^r) \in \mathscr{L}^X = \Sigma^X \times (Diff^3([0,1]))^3. \quad (3.2)$$

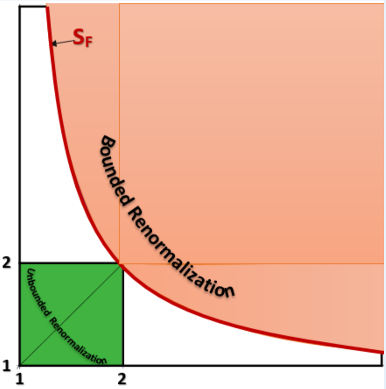
$$f(x) = \begin{cases} (1-x_2)q_s \circ \varphi\left(\frac{x_1 - x}{x_1}\right) + x_2 & \text{if} \quad x \in [x_1 \sim 1, 0[\\ x_1\left(\varphi^l(\frac{x_3 - x}{x_3})\right)^{\ell_1} & \text{if} \quad x \in]0, x_3] \\ 0 & \text{if} \quad x \in [x_3, x_4] \\ x_2\left(\varphi^r(\frac{x - x_4}{1 - x_4})\right)^{\ell_2} & \text{if} \quad x \in [x_4, 1 \sim x_1]; \end{cases} \quad (3.3)$$

$$q_s(x) = [(1-s)x + s]^{\ell_2}\left[1 - s^{\ell_2}\right]^{-1}. \quad (3.4)$$

Our Contributions

$$\text{if } 0 < x_2 < x_3, \text{ then } R(f) := h \circ P_{re} R(f) h^{-1} \in \mathscr{L}^X. \qquad (3.5)$$

$$\ell_1 + \ell_2 + 1 + \sqrt{(\ell_1 - \ell_2)^2 + 2(\ell_1 + \ell_2) + 1} - 2\ell_1\ell_2 < 0 \qquad I_{1,1}(\ell_1, \ell_2)$$



Theorem 2 (B. TANGUE NDAWA (2024))

1. $[\infty - R]$: $(\ell_1, \ell_2) \in (1,2)^2$ *with **(A-DG)**;*
2. $[B - R]$: *if* $I_{1,1}(\ell_1, \ell_2)$ *holds and additional condition.*

Proposition 3

$f \in \mathscr{L}_0^X$, $P_{re}R(f)$ defined on $[x_1, x_2)$.

$$h : x \in [x_1, x_2) \longmapsto \frac{x}{x_1} \in [x_2/x_1, 1) \quad (3.6)$$

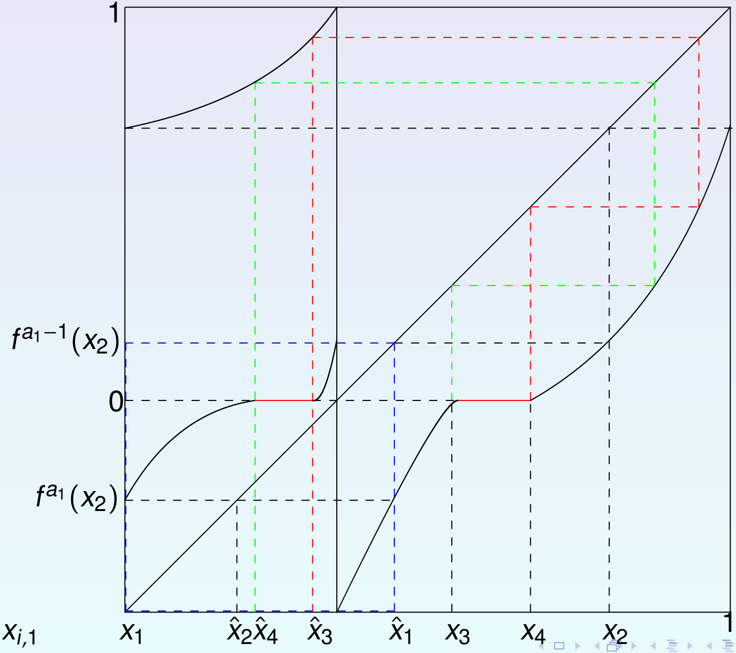
then,

$$Rf := h \circ P_{re}R(f) \circ h^{-1} \in \mathscr{L}^X; \quad (3.7)$$

that is,

$$Rf := (x_{1,1}, x_{2,1}, x_{3,1}, x_{4,1}, s_1, \varphi_1, \varphi_1^l, \varphi_1^r) \in \mathscr{L}^X; \quad (3.8)$$

where,



$\hat{x}_i = x_1 x_{i,1}$

$$
\begin{aligned}
x_{1,1} &= \frac{x_2}{x_1}, \\
x_{2,1} &= \left(\varphi^l\left(\frac{x_3 - x_2}{x_3}\right)\right)^{\ell_1}, \\
x_{3,1} &= 1 - \varphi^{-1} \circ q_s^{-1}\left(\frac{x_4 - x_2}{1 - x_2}\right), \\
x_{4,1} &= 1 - \varphi^{-1} \circ q_s^{-1}\left(\frac{x_3 - x_2}{1 - x_2}\right), \\
s_1 &= \varphi^l\left(\frac{x_3 - x_2}{x_3}\right), \\
\varphi_1 &= Z_{[1-x_2/x_3,1]}(\varphi^l), \\
\varphi_1^l &= \varphi^r \circ Z_{[1-x_{3,1},1]}(q_s \circ \varphi), \\
\varphi_1^r &= Z_{[0,1-x_2/x_3]}(\varphi^l)(q_s \circ \varphi) \circ Z_{[0,1-x_{4,1}]}(q_s \circ \varphi).
\end{aligned}
$$

Proof of main result: Changes of Variables

$(X) \longleftrightarrow (S)$

$$S_1 = \frac{x_3 - x_2}{x_3},\ S_2 = \frac{1 - x_4}{1 - x_2},\ S_3 = \frac{x_3}{1 - x_4},\ S_4 = -\frac{x_2}{x_1},\ S_5 = s; \qquad (3.9)$$

$$\begin{cases} x_1 &= -\dfrac{S_3(1 - S_1)S_2}{(1 + S_3(1 - S_1)S_2)S_4} \\ x_2 &= \dfrac{S_3(1 - S_1)S_2}{1 + S_3(1 - S_1)S_2} \\ x_3 &= \dfrac{S_3 S_2}{1 + S_3(1 - S_1)S_2} \\ x_4 &= 1 - \dfrac{S_2}{1 + S_3(1 - S_1)S_2} \end{cases} \qquad (3.10)$$

$(S) \longleftrightarrow (Y)$

$$y_1 = S_1,\ y_2 = \ln S_2,\ y_3 = \ln S_3,\ y_4 = \ln S_4,\ y_5 = \ln S_5. \qquad (3.11)$$

Proof of main result: Super Formula

$$w_n(f) := (y_{i,n} := y_i(R^n f))_{i=2,3,4,5} \tag{3.12}$$

Proposition 4 (B. TANGUE NDAWA (2024))

Let $(\ell_1, \ell_2) \in (1,2)^2$. Then, there exists $\lambda_u > 1$, $\lambda_s \in (0,1)$ E^u, E^s, E^+, $w_{fix} \in \mathbb{R}^4$, s.t, $f \in \mathcal{W}_c$ with critical exponents (ℓ_1, ℓ_2), there exists $c_u(f), c'_u(f) < 0$, $c_s(f), c'_s(f)$, $c_+(f)$ and $c'_+(f)$ s.t $\forall n := 2p_n \in \mathbb{N}^*; p_n \in \mathbb{N}$,

$$w_n(f) = c_\varsigma(f)\lambda_\varsigma^{p_n} E^\varsigma + w_{fix} + O(e, c_u(f), \lambda_u, n) \tag{3.13}$$

$$w_{n+1}(f) = c'_\varsigma(f)\lambda_\varsigma^{p_n} E^\varsigma + w_{fix} + O(e, c'_u(f), \lambda_u, n); \tag{3.14}$$

$$O(e, c_u(f), \lambda_u, n) \equiv O\left(\left(e^{c_u(f)\lambda_u^{p_n - 4}}\right)^{1/\overline{\ell}}\right); \ \overline{\ell} := \max\{\ell_1, \ell_2\}. \tag{3.15}$$

Proof of main result: Super Formula

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Proposition 4 (B. TANGUE NDAWA (2024))

Let $(\ell_1, \ell_2) \in (1,2)^2$. *Then, there exists* $\lambda_u > 1$, $\lambda_s \in (0,1)$ E^u, E^s, E^+, $w_{fix} \in \mathbb{R}^4$, *s.t,* $f \in \mathcal{W}_c$ *with critical exponents* (ℓ_1, ℓ_2), *there exists* $c_u(f), c'_u(f) < 0$, $c_s(f), c'_s(f)$, $c_+(f)$ *and* $c'_+(f)$ *s.t* $\forall n := 2p_n \in \mathbb{N}^*; p_n \in \mathbb{N}$,

$$w_n(f) = c_\varsigma(f)\lambda_\varsigma^{p_n} E^\varsigma + w_{fix} + O(e, c_u(f), \lambda_u, n) \tag{3.13}$$

$$w_{n+1}(f) = c'_\varsigma(f)\lambda_\varsigma^{p_n} E^\varsigma + w_{fix} + O(e, c'_u(f), \lambda_u, n); \tag{3.14}$$

$$O(e, c_u(f), \lambda_u, n) \equiv O\left(\left(e^{c_u(f)\lambda_u^{p_n - 4}}\right)^{1/\bar{\ell}}\right); \; \bar{\ell} := \max\{\ell_1, \ell_2\}. \tag{3.15}$$

And, we have also

$$dist(\varphi_n) = O\left(e^{\frac{c_u(f)\lambda_u^{p_{n-2}}}{\ell_1}}\right)$$
$$dist(\varphi_n^l) = O\left(e^{\frac{c_u(f)\lambda_u^{p_{n-1}}}{\ell_2}}\right) \quad (3.16)$$
$$dist(\varphi_n^r) = O\left(e^{\frac{c_u(f)\lambda_u^{p_n}}{\ell_1}}\right)$$

with

$$Dist(\varphi_{|I}) = \sup_{x,y\in I} \log \frac{Df(y)}{Df(x)} \quad (3.17)$$

Proposition 5 (B. TANGUE NDAWA (2024))

Let $f \in \mathcal{W}_c$. For every n,

$$dist(\varphi_n^l) = O\left(\alpha_{n-1}^{\frac{1}{\ell_1}, \frac{1}{\ell_2}}\right), \; dist(\varphi_n) = O\left(\alpha_{n-2}^{\frac{1}{\ell_1}, \frac{1}{\ell_2}}\right), \; dist(\varphi_n^r) = O\left(\alpha_n^{\frac{1}{\ell_1}, \frac{1}{\ell_2}}\right)$$

Lemma 6

Let $f \in \mathcal{W}_c$, then

1. $\dfrac{x_{2,n+1}}{x_{1,n}} = O(\alpha_{n+1})$,
2. $\dfrac{x_{3,n+1}}{x_{1,n}} = O(\alpha_{n+1})$,
3. $\dfrac{x_{1,n} - x_{4,n+1}}{x_{1,n}} = O(\alpha_n)$,
4. $S_{1,n} = \dfrac{x_{3,n} - x_{2,n}}{x_{3,n}} = O(\alpha_{n+1})$.
5. $s_n = S_{5,n} = O(\alpha_n)$.

Proposition 7 (B. TANGUE NDAWA (2024))

$f \in \mathcal{W}$, $n \in \mathbb{N}$ and

$$R^n(f) = \left(S_{1,n}, S_{2,n}, S_{3,n}, S_{4,n}, S_{5,n}, \varphi_n, \varphi_n^l, \varphi_n^r\right). \tag{3.18}$$

If n is even, then

$$\begin{array}{llcl}
1. & S_{1,n+1} & = & 1 - \frac{\ell_2 S_{1,n}^{\ell_1}}{S_{2,n}}\left(1 + O\left(\alpha_{n-1}^{1/\ell_2}\right)\right), \\
2. & S_{2,n+1} & = & \frac{S_{1,n}S_{2,n}S_{3,n}}{\ell_2 S_{5,n}^{\ell_2-1}}\left(1 + O\left(\alpha_{n-2}^{1/\ell_1}\right)\right), \\
3. & S_{3,n+1} & = & \frac{S_{5,n}^{\ell_2-1}}{S_{1,n}S_{3,n}}\left(1 + O\left(\alpha_{n-2}^{1/\ell_1}\right)\right), \\
4. & S_{4,n+1} & = & \frac{S_{1,n}^{\ell_1}}{S_{4,n}}\left(1 + O\left(\alpha_{n-1}^{1/\ell_2}\right)\right), \\
5. & S_{5,n+1} & = & S_{1,n}\left(1 + O\left(\alpha_{n-1}^{1/\ell_2}\right)\right).
\end{array} \tag{3.19}$$

Corollary 8 (B. TANGUE NDAWA (2024))

$f \in \mathcal{W}_c$, $n \in \mathbb{N}$ and

$$R^n(f) = \left(S_{1,n}, S_{2,n}, S_{3,n}, S_{4,n}, S_{5,n}, \varphi_n, \varphi_n^l, \varphi_n^r\right). \tag{3.20}$$

Then,

$$\frac{\ell_2 S_{1,2n}^{\ell_1}}{S_{2,2n}} = 1 + O\left(\alpha_{2n-1}^{\frac{1}{\ell_2}}\right) \quad \text{and} \quad \frac{\ell_1 S_{1,2n+1}^{\ell_2}}{S_{2,2n+1}} = 1 + O\left(\alpha_{2n}^{\frac{1}{\ell_1}}\right). \tag{3.21}$$

Proposition 9 (B. TANGUE NDAWA (2024))

$f \in \mathcal{W}_c$. For $n \in \mathbb{N}$ even, the following equality holds.

$$w_{n+2} = L_{(\ell_2,\ell_1)} w_{n+1} + w^*_{(\ell_1,\ell_2)} + \underline{O}(n, \ell, \alpha). \tag{3.22}$$

Let w_{fix} be the fixed point of the equation $\overline{L}w + \overline{w}^*_{(\ell_1,\ell_2)} = w$

Lemma 10 (B. TANGUE NDAWA (2024))

For every p_n large enough,

$$w_{2p_n}(f) = c_u(f)\lambda_u^{p_n}E^u + c_s(f)\lambda_s^{p_n}E^s + c_+(f)E^+ + w_{fix} + O(n,\ell_1,\overline{\ell},\alpha_0);$$

$$w_{2p_n+1}(f) = c'_u(f)\lambda_u^{p_n}E^u + c'_s(f)\lambda_s^{p_n}E^s + c'_+(f)E^+ + w_{fix} + O(n,\ell_2,\overline{\ell},\alpha_0);$$

where, $O(n,\ell_1,\overline{\ell},\alpha_0)$ resp $O(n,\ell_2,\overline{\ell},\alpha_0)$ is the vector whose components are equal to

$$0\left(\alpha_0^{\left(\frac{2}{\ell_1}\right)^{p_{n-4}}}\right)^{\frac{1}{\ell}} \quad resp \quad 0\left(\alpha_0^{\left(\frac{2}{\ell_2}\right)^{p_{n-4}}}\right)^{\frac{1}{\ell}}. \tag{3.23}$$

Lemma 11 (B. TANGUE NDAWA (2024))

Let $f \in \mathcal{W}_c$, then,

$$\alpha_{2p_n} = O\left(e^{c_u(f)\lambda_u^{pn}(e_2^u+e_3^u)}\right) \text{ with } c_u(f) < 0 \quad (3.24)$$

and

$$\alpha_{2p_n+1} = O\left(e^{c'_u(f)\lambda_u^{pn}(e_2^u+e_3^u)}\right) \text{ with } c'_u(f) < 0. \quad (3.25)$$

Corollary 12 (B. TANGUE NDAWA (2024))

$$
\begin{aligned}
e^{\Psi_{1,n}} &= \frac{S_{2,n}}{\ell_n(\varphi_n^{-1}\circ q_{s_n}^{-1}(1-S_{2,n}))}\cdot\frac{\left(\varphi_n^{l}(S_{1,n})\right)^{\ell_{n+1}}}{S_{1,n}^{\ell_{n+1}}},\\
e^{\Psi_{2,n}} &= \frac{\ell_n s_{5,n}^{\ell_n-1}\varphi_n^{-1}\circ q_{s_n}^{-1}(S_{1,n}S_{2,n}S_{3,n})}{S_{1,n}S_{2,n}S_{3,n}}\cdot\frac{1}{1-\varphi_n^{l}(S_{1,n})},\\
e^{\Psi_{3,n}} &= \frac{S_{1,n}S_{3,n}(1-\varphi_n^{-1}\circ q_{s_n}^{-1}(1-S_{2,n}))}{S_{5,n}^{\ell_{2,n}-1}\varphi_n^{-1}\circ q_{s_n}^{-1}(S_{1,n}S_{2,n}S_{3,n})},\\
e^{\Psi_{4,n}} &= \left(\frac{\varphi_n^{l}(S_{1,n})}{S_{1,n}}\right)^{\ell_{n+1}}, e^{\Psi_{5,n}}\frac{\varphi_n^{l}(S_{1,n})}{S_{1,n}}, \Psi_{0,n} = \frac{S_{1,n}^{\ell_{n+1}}}{S_{2,n}}
\end{aligned}
\tag{3.26}
$$

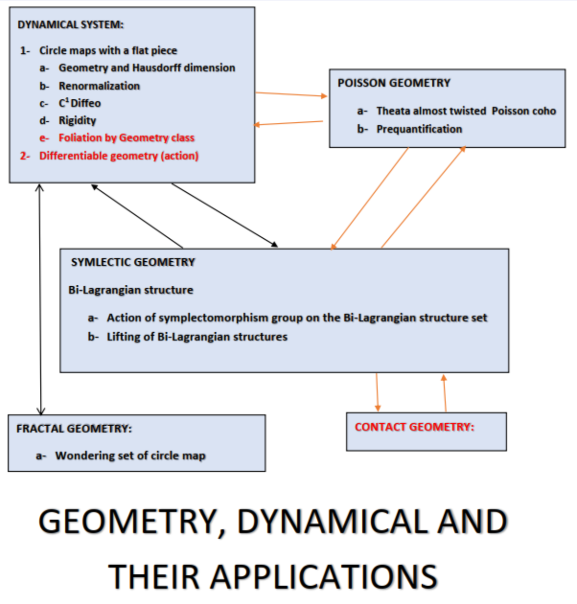
If the sequences in ***(3.26)*** *are uniformly bounded, then*

1. $w_n(f)$ *tends to infinity exponentially when* $(\ell_1,\ell_2)\in[1,2]^2\setminus\{(2,2)\}$
2. $w_n(f)$ *is bounded when* (ℓ_1,ℓ_2) *verifies the inequality* $I_{1,1}(\ell_1,\ell_2)$.

Lemma 13 (B. TANGUE NDAWA (2024))

Let $(\ell_1, \ell_2) \in (1,2)^2$ *and* $f \in \mathcal{W}_c$ *with critical exponent* (ℓ_1, ℓ_2). *Then for* $n := 2p_n$,

$$\begin{aligned}
-x_{1,n} &= e^{c_u(f)\lambda_u^{p_n}(e_2^u+e_3^u-e_4^u)+c_s(f)\lambda_s^{p_n}(e_2^s+e_3^s-e_4^s)-c_+}+0((e^{c_u(f)\lambda_u^{p_{n-4}}(e_2^u+e_3^u)})^{1/\bar{\ell}}) \\
x_{2,n} &= e^{c_u(f)\lambda_u^{p_n}(e_2^u+e_3^u)+c_s(f)\lambda_s^{p_n}(e_2^s+e_3^s)}+0((e^{c_u(f)\lambda_u^{p_n}(e_2^u+e_3^u)})^{1/\bar{\ell}}) \\
x_{3,n} &= e^{c_u(f)\lambda_u^{p_n}(e_2^u+e_3^u)+c_s(f)\lambda_s^{p_n}(e_2^s+e_3^s)}+0((e^{c_u(f)\lambda_u^{p_{n-4}}(e_2^u+e_3^u)})^{1/\bar{\ell}}) \\
1-x_{4,n} &= e^{c_u(f)\lambda_u^{p_n}e_2^u+c_s(f)\lambda_s^{p_n}e_2^s}+0((e^{c_u(f)\lambda_u^{p_{n-4}}(e_2^u+e_3^u)})^{1/\bar{\ell}})
\end{aligned}$$


DYNAMICAL SYSTEM:
1- Circle maps with a flat piece
a- Geometry and Hausdorff dimension
b- Renormalization
c- C¹ Diffeo
d- Rigidity
e- Foliation by Geometry class
2- Differentiable geometry (action)
POISSON GEOMETRY
a- Theata almost twisted Poisson coho
b- Prequantification
SYMLECTIC GEOMETRY
Bi-Lagrangian structure
a- Action of symplectomorphism group on the Bi-Lagrangian structure set
b- Lifting of Bi-Lagrangian structures
FRACTAL GEOMETRY:
a- Wondering set of circle map
CONTACT GEOMETRY:
GEOMETRY, DYNAMICAL AND
THEIR APPLICATIONS

Merci Brazzaville